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Turbulent dispersion from sources near two-dimensional obstacles

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The first part of the paper is an extension of the statistical theory of turbulent diffusion to the problem of turbulent diffusion from a point or line source in an inhomogeneous straining flow such as flow round a two-dimensional body. The new effects which have to be considered are the convergence and divergence of mean streamlines, the inhomogeneity of the mean and turbulent velocities, and the presence of a boundary. We assume that the upstream turbulence intensity $u'_\infty/\overline{u}_\infty$ is weak, i.e. $u'_\infty/\overline{u}_\infty \ll 1$, and that molecular diffusion is negligible, i.e. $(\overline{u}_\infty/u'_\infty)^2\,[D/(a\overline{u}_\infty)] \ll 1$, D and a being the molecular diffusivity and the scale of the obstacle respectively. The theory predicts the mean-square dispersion of the plume about the mean streamline through the source in terms of the Lagrangian statistics of the turbulence. Making the further assumption that the scale L_E of the turbulence is large enough to satisfy the condition that $(u'_\infty/\overline{u}_\infty)\,(a/L_E) \ll 1$, it is shown that the turbulent dispersion can be calculated in terms of the Eulerian statistics, which can either be measured or in some cases calculated. In the second part we analyse diffusion from various sources in potential flows over two-dimensional obstacles assuming constant (or variable) eddy diffusivity, and compare the results with those of the more rigorous statistical analysis for sources one or two diameters upwind of the obstacle. However, unlike the statistical analysis, this eddy-diffusivity analysis can also be extended to calculate diffusion from sources placed some distance upwind of an obstacle, and an example is given of how this analysis may be applied to calculating concentrations on hills due to distant sources.

1. Introduction

The effect of the wind in dispersing pollution from sources above level ground has been extensively studied. An excellent recent review is that of Pasquill (1971). But there have been very few studies of how airborne pollution is dispersed by the wind in the more complicated flows that occur in the presence of buildings at one end of the scale and of hills at the other end. This is a problem of pressing importance to architects, planners and public health officials for obvious reasons (Holford 1971; McCormick 1971). There have been extensive experimental investigations of pollution concentrations on models placed in wind tunnels with sources placed in various positions near the models, but

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usually the mean and turbulent velocities were not measured in any detail (Halitsky 1968); there have been some detailed measurements of pollution concentrations downwind of buildings when wind velocities were also carefully measured (Halitsky 1969). Another full-scale experiment was recently reported by Rodliffe & Fraser (1971), for these problems. Theoretical models, using a diffusion equation, have been developed by Burger (1964) and Stumke (1964) in the course of several papers, but with a number of unsatisfactory assumptions and with little reference to the physical processes of turbulent diffusion. A similar approach has been used by Berlyand and his colleagues in Leningrad (Berlyand 1972). A simpler and more realistic model was developed by Scriven & Moore (1962) for diffusion in the wake of a bluff body and variants on it were developed by Rodliffe & Fraser (1971)†, but no theoretical models have yet been developed to predict the kind of measurements made by Halitsky and others.

The first aim of this paper (§ 2) is to apply Taylor's (1921) statistical approach to the problem of turbulent diffusion from point sources placed in a flow over a two-dimensional body. (Taylor's theory has been further confirmed by the recent definitive experiments of Snyder & Lumley (1971).) It is not sufficient just to consider matter being carried along a streamline and diffusing away from it, as if the flow were uniform, because there are two additional effects. The first is that streamlines converge and diverge, thus reducing or amplifying the dispersion. The second is that the turbulence itself varies along each streamline because of the distortion of the turbulence by the mean flow and the 'blocking' of the turbulence by the body. Despite these complications we eventually find that the mean-square displacement of a fluid particle is given by an integral similar to that found by Taylor. Using this integral and the results of Hunt's (1973) analysis of the turbulence round a bluff body, we can then examine the effect of varying the integral scale and intensity of the turbulence on diffusion from point sources placed at various positions upwind of a cylinder.

In § 3 we use existing solutions and develop some new solutions to the diffusion equation

$$U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right),$$

where D is either the molecular diffusivity or an eddy diffusivity, to calculate diffusion from line and point sources situated in flows around two-dimensional obstacles. The advantage of using this equation is that the effects of the boundary condition $\partial C/\partial n = 0$ are more easily calculable than by means of the statistical approach. We then use the solutions to compare the results of problems analysed by the statistical and diffusion-equation approaches. We also analyse the problem of dispersion from a source placed on the body, which cannot yet be tackled by statistical methods. In § 4 we draw some general conclusions about the validity of the two methods and we discuss the applicability of our simplified analyses to the real problems of pollution in atmospheric conditions.

† Some comments on their theory were made by Hunt (1971).

2. Statistical analysis

2.1. *General results*

The dispersion of matter from a point or line source placed in a turbulent stream can be studied by calculating the displacement $\mathbf{X}$ caused by fluid motions and molecular motion (i.e. diffusion) of the ensemble of all particles which have passed through the source P. Let P be situated in a turbulent flow around a cylindrical obstacle, the axis of which is parallel to the z axis. (Much of the analysis is also valid for any turbulent flow which is homogeneous in the z direction, such as a shear flow, but this will not be considered in detail.)

Assume that the turbulence† is homogeneous in the z direction and that at each point in the flow turbulent velocities are stationary random functions of time t. Then the fluid velocity $\mathbf{u}^*$ can be expressed as

$$\mathbf{u}^* = \mathbf{U} + \mathbf{u}, \tag{2.1}$$

where

$$\mathbf{U} = \overline{\mathbf{u}^*} = \lim_{\tau \to \infty} \left(\frac{1}{\tau} \int_0^\tau \mathbf{u}^*(t)\, dt \right)$$
$$= (U(x,y), V(x,y), 0). \tag{2.2}$$

All relevant properties of the velocity field are assumed to be known, the problem being to calculate the dispersion from these properties. As a reference for measuring the dispersion from the source it is convenient to define the 'mean streamline' $y_s(x)$ as the streamline of the mean motion which passes through the source P, at $(x_p, y_p, 0)$. $y_s(x)$ is calculated from the equation

$$dy_s/dx = V(x, y_s)/U(x, y_s) \quad (x > x_p), \tag{2.3}$$

with the initial condition $y_s(x_p) = y_p$. As we shall see later, $y_s(x)$ is *not* the same as the mean particle path.

Co-ordinates and velocity components in the x, y plane parallel to and normal to this streamline are defined as (s, n) and $(\tilde{U}, \tilde{V})$, $(\tilde{u}, \tilde{v})$ (figure 1). We denote the value of a parameter on the mean streamline $n = 0$ by the suffix zero, e.g. $\tilde{U}(s, 0) = \tilde{U}_0(s)$. For functions on $z = 0$, or for functions which are independent of z we shall in general omit the z co-ordinate.

If the turbulent velocity is sufficiently large or the distance from the source is not large, then molecular diffusion is negligible, which implies that $\mathbf{X}$ is determined solely by the mean and turbulent motion of infinitesimal fluid particles and not by relative motion of molecules. The errors caused by this assumption will be discussed *a posteriori* using the analysis of Saffman (1960).

If (N, Z) is the displacement of any fluid particle in a plane normal to the mean streamline produced by the mean and turbulent fluid motions, then N and Z are random functions of (s, z, t) and (s, n, t) respectively. But if we follow the path of a fluid particle which passes through the source at time t_0‡ then N and Z are

† This turbulence may result from the incident turbulence (which is rotational) or the turbulence and vortex shedding of the wake (which is irrotational); Hunt (1973).

‡ t_0 being, for example, the time from the beginning of one of an ensemble of similar experiments.

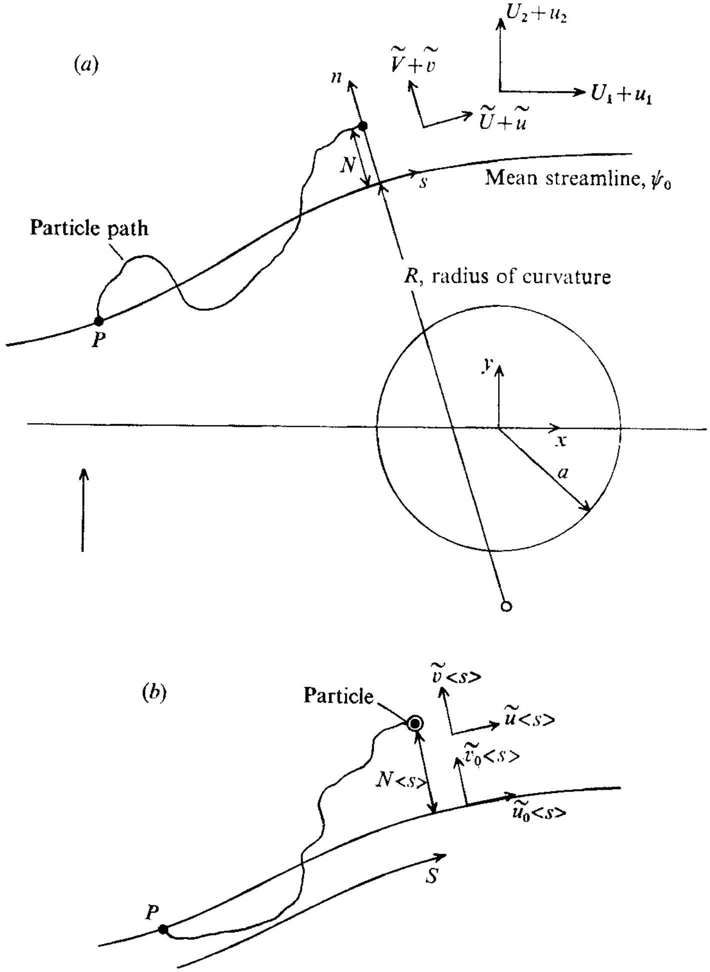


FIGURE 1(*a*). Obstacles, mean streamline and co-ordinate system. (*b*) Actual and approximate turbulent velocity of a fluid particle at the point (S, N).

random functions of $T = t - t_0$, denoted by $N\langle T\rangle$ and $Z\langle T\rangle$. (Angular brackets denote the value of a particle's displacement or velocity expressed in terms of the particle's time of travel T or displacement s.) By definition of a 'fluid particle', $N\langle T\rangle$ is given in terms of the fluid velocity by the Lagrangian equation

$$dN/dT = \tilde{V}\langle T\rangle + \tilde{v}\langle T\rangle, \quad dZ/dT = w\langle T\rangle, \tag{2.4}$$

where $\tilde{V}\langle T\rangle$ and $\tilde{v}\langle T\rangle$ are the components of the mean and turbulent velocity normal to the mean streamline, and w is the component of the turbulent velocity in the z direction when the particle has travelled for a time T from the source. Note that $\tilde{V}$ is caused by the convergence and divergence of the streamlines. The problem, as always, is to relate $\tilde{V}\langle T\rangle$ and $\tilde{v}\langle T\rangle$, say, to the velocities measured in fixed frames of reference, which when the particle is at (s, N, z) at time $T + t_0$ are $\tilde{V}(s, N)$ and $\tilde{v}(s, N, z, T + t_0)$. This requires making the assumption that the r.m.s. turbulent velocities are everywhere small compared with mean velocities. Therefore we assume that the upstream turbulence is weak and that the mean streamline is sufficiently far from the intense turbulence in the wake of the body. Thus

$$u'_\infty/U_\infty = \alpha \ll 1, \tag{2.5}$$

where $u'_\infty = (\overline{u_1^2})^{\frac{1}{2}}$ and $U_\infty = U$ as $x \to -\infty$. The assumption (2.5) can be shown *a posteriori* to imply that, if the ensemble averages of $N^2\langle s\rangle$ and $Z^2\langle s\rangle$ are

$$E[N^2\langle s\rangle] \quad \text{and} \quad E[Z^2\langle s\rangle],$$

then

$$\{E[N^2\langle s\rangle]\}^{\frac{1}{2}}/a = \delta \ll 1, \tag{2.6}$$

in other words that the average distance moved from the mean streamline by a particle, i.e. the width of the plume, is small compared with the distance over which the mean velocity changes. It also follows from (2.5) that, for all except a tiny proportion of fluid particles which have large turbulent velocities and the paths of which may include closed loops, $N\langle s\rangle$, $Z\langle s\rangle$ and $T\langle s\rangle$ for any particle are single-valued random functions of s. Thence for any particle, N, $\tilde{v}$, etc., can be expressed unambiguously as functions of s or T, e.g.

$$N\langle T\rangle = N\langle s\rangle.$$

Since

$$T\langle s\rangle = \int_0^s \frac{ds'}{\bar{U}(s', N(s')) + \tilde{u}(s', N(s'), Z(s'), T(s') + t_0)},$$

following Batchelor & Townsend (1956), this expression can be expanded in a Taylor series as

$$\begin{aligned} T\langle s\rangle &= \int_0^s \frac{ds'}{\bar{U}_0(s')}\left\{1 - \frac{N}{\bar{U}_0(s')}\left(\frac{\partial \bar{U}}{\partial n}\right)_{n=0} - \frac{\tilde{u}_0}{\bar{U}_0(s')}(s', T_0(s') + t_0)\ldots\right\} \\ &= T_0(s)\{1 + O(\delta) + O(\alpha)\}, \end{aligned} \tag{2.7}$$

where

$$T_0\langle s\rangle = \int_0^s \frac{ds'}{\bar{U}_0(s')},$$

which is the time taken to travel along the mean streamline from P to $(s, 0)$. Thus, although $T\langle s\rangle$ is a random function, to a first approximation it is equal to the determinate function $T_0(s)$. It also follows from N being a single-valued function of s that

$$\begin{aligned} \frac{dN}{dT}\langle T\rangle &= (\bar{U} + \tilde{u})\frac{dN}{ds}\langle s\rangle \\ &= \bar{U}_0\langle s\rangle \frac{dN}{ds}\langle s\rangle + \left[\tilde{u}\langle s\rangle + N\frac{\partial \bar{U}}{\partial n}(s, 0)\right]\frac{dN}{ds} + \ldots. \end{aligned} \tag{2.8}$$

In order to calculate $N\langle s\rangle$, the relations for $\bar{V}\langle T\rangle$ and $\tilde{v}\langle T\rangle$ in (2.4) can be expressed in terms of velocities and their derivatives on the mean streamline by equating the velocity of a particle at time T after leaving the source to the local velocity at (s, N, z) at time $T + t_0$. Using Taylor's series

$$\bar{V}\langle T\rangle = \bar{V}(s, N) = N(\partial \bar{V}/\partial n)(s, n = 0) + N^2(\partial^2 \bar{V}/\partial n^2)_{n=0} + \ldots, \tag{2.9}$$

since $\tilde{V} = 0$ at $N = 0$, by definition. But by the continuity at the point (s, n, z)

$$\frac{\partial \tilde{V}}{\partial n} + \frac{\tilde{V}}{R+n} + \frac{\partial \tilde{U}}{\partial s} = 0,$$

where R is the radius of curvature of the mean streamline at $n = 0$. Therefore at $n = 0$,

$$\partial \tilde{V}/\partial n = -d\tilde{U}_0(s)/ds. \tag{2.10}$$

Since

$$\tilde{v}\langle T\rangle = \tilde{v}\langle s\rangle = \tilde{v}\,(s, N, z, T+t_0),$$

$$\tilde{v}\langle T\rangle = \tilde{v}(s,0,T_0+t_0) + (T-T_0)\,(\partial\tilde{v}/\partial t)\,(s,0,T_0+t_0) + Z(\partial\tilde{v}/\partial z)\,(s,0,T_0+t_0) + N(\partial\tilde{v}/\partial n)\,(s,0,T_0+t_0)\ldots. \tag{2.11}$$

The order of magnitude of the second and third terms depends on the scale of the turbulence. If the (Eulerian) integral scale of the turbulence upstream is defined, in a fixed co-ordinate system, as

$$L_E = \int_0^\infty \overline{u_{\infty 1}(x)\,u_{\infty 1}(x+r_x)}\,dr_x/\overline{u^2_{\infty 1}},$$

then

$$\left.\begin{aligned} (T-T_0)\,\partial\tilde{v}/\partial t &= O(\alpha^2 T_0 U_\infty^2/L_E), \quad O(\alpha^2 T_0 U_\infty^2/a) \\ \left.\begin{aligned} Z(\partial\tilde{v}/\partial z) \\ N(\partial\tilde{v}/\partial n)\end{aligned}\right\} &= O(U_\infty \alpha a\delta/L_E), \quad O(\alpha U_\infty \delta)\end{aligned}\right\} \text{ for } L_E \lessgtr a, \text{ respectively.}$$

Since $T_0 = O(\alpha/U_\infty)$, it follows from (2.11) that

$$\tilde{v}\langle s\rangle = \tilde{v}_0\langle s\rangle\{1+[O(\alpha a/L_E), O(\alpha)] + [O(\alpha a\delta/L_E), O\,(\alpha\delta)]\}. \tag{2.12}$$

for $L_E \lessgtr a$, respectively.

It now follows from (2.8)–(2.12) that (2.4) can be expressed either as

$$\tilde{U}_0\langle s\rangle\,(dN/ds)\,\langle s\rangle + N(d\tilde{U}_0/ds)\,\langle s\rangle - \tilde{v}\langle s\rangle = \Delta_1, \tag{2.13a}$$

where $\tilde{v}\langle s\rangle$ is the turbulent component of the velocity of the actual fluid particle, which may or may not be on the mean streamline, and Δ_1 is an error term, or as

$$\tilde{U}_0\langle s\rangle\,(dN/ds)\,\langle s\rangle + N(d\tilde{U}_0/ds)\,\langle s\rangle - \tilde{v}_0\langle s\rangle = \Delta_2, \tag{2.13b}$$

where $\tilde{v}_0\langle s\rangle$ is the turbulent component of the velocity normal to the mean streamline at $n = 0$ when the fluid particle is at a distance s along the streamline and displaced a distance N normal to the streamline, i.e. $\tilde{v}_0\langle s\rangle$ is only approximately equal to the actual velocity of the fluid particle. See figure 1 (*b*).

The object of expressing (2.13) in two ways is because of the error terms. It follows from (2.7)–(2.10) that

$$\Delta_1 = \{-[\tilde{u}(s,0,z,T_0+t_0) + N(d\tilde{U}/dn)\,(s,0)]\,(dN/ds) + N^2(\partial^2\tilde{V}/\partial n^2)\,(s,0)\} \times (1+O(\delta)+O(\alpha)),$$

whence

$$\Delta_1 = U_\infty\{O(\alpha\delta)+O(\delta^2)\}. \tag{2.14a}$$

But from (2.7)–(2.12),

$$\Delta_2 = \Delta_1 + (T - T_0)\,(\partial\tilde{v}/\partial t)\,(s, 0, T_0 + t_0) + N(\partial\tilde{v}/\partial n)(s, 0, T_0 + t_0) + Z\,(\partial\tilde{v}/\partial z)\,(s, 0, T_0 + t_0),$$

whence, taking a typical value of $T_0 = a/U_\infty$,

$$\Delta_2 = U_0[O(\alpha\delta) + O(\delta^2) + O(\alpha^2 a/L_E, \alpha^2) + O(\alpha\delta a/L_E, \alpha\delta)], \qquad (2.14b)$$

depending on whether

$$L_E \lesseqgtr a.$$

Thus if the Eulerian integral scale L_E is of the order of or greater than the typical scale of the body, then since $\delta = O(\alpha)$ (as shown later) Δ_2 is comparable with Δ_1 and (2.13*b*) is as accurate as (2.13*a*). If the scale of the incident turbulence is small compared with the size of the body, and yet $\alpha a/L_E \ll 1$, then (2.13*b*) is still valid, but (2.13*a*) is the more accurate of the two. But in the extreme limit $\alpha \to 0$, $\delta \to 0$, $\alpha a/L_E = O(1)$ then (2.13*b*) is no longer valid, but the error term (2.14*a*) remains small and consequently (2.13*a*) is still valid.

The solution to (2.13*a*) subject to Δ_1 being negligible is

$$N\langle s\rangle = \frac{1}{\bar{U}_0\langle s\rangle}\int_0^s \tilde{v}\langle s'\rangle\, ds', \qquad (2.15a)$$

and the solution to (2.13*b*) subject to Δ_2 being negligible is

$$N\langle s\rangle = \frac{1}{\bar{U}_0\langle s\rangle}\int_0^s \tilde{v}_0\langle s'\rangle\, ds'. \qquad (2.15b)$$

These results can be expressed in terms of fixed co-ordinates (s, N, Z, t), where $t = T_0(s) + t_0$, as

$$N(s, T_0(s) + t_0) = \frac{1}{\bar{U}_0(s)}\int_0^s \tilde{v}(s', N(s', T_0(s') + t_0), Z(s', T_0(s') + t_0), T_0(s') + t_0)\, ds' \qquad (2.16a)$$

and

$$N(s, T_0(s) + t_0) = \frac{1}{\bar{U}_0(s)}\int_0^s \tilde{v}_0(s', T_0(s') + t_0)\, ds'. \qquad (2.16b)$$

The physical interpretation of these solutions is that the random displacement $N\langle s\rangle$ normal to the mean streamline of particles leaving P is proportional to the integral with respect to the mean-streamline co-ordinate of the turbulent velocity component of the fluid particle normal to the mean streamline. Alternatively N can be thought of as the solution of an integral equation which involves the velocity at given points at different times, equation (2.16*a*). However, if the scale of the turbulence is not very small N is approximately given by the integral of $\tilde{v}$ experienced by a point moving along the mean streamline at the mean velocity, equation (2.15*b*). This can be expressed in terms of $\tilde{v}$ on the streamline at specified values of s and t, equation (2.16*b*).

Consider the mean-square displacement. By the ergodic theorem, since the turbulence is assumed to be a stationary random function of time (in fixed co-ordinates) the ensemble and time averages of N^2 are identical. The ensemble average of N^2 for a particle leaving the source in a number of experiments is identical to the time average $\overline{N^2}(s)$ of N^2, for a continuous stream of particles leaving P:

$$E[N^2\langle s\rangle] = \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T N^2(s, T_0(s)+t_0)\,dt_0\right\} = \overline{N^2}(s).$$

Therefore from (2.15*a*) in the limit $\alpha \to 0$, $\delta \to 0$

$$\overline{N^2}(s) = \frac{1}{\overline{U}_0^2(s)}\int_0^s\int_0^s \rho_{\tilde{v}}\langle s' s''\rangle\, ds'\, ds'', \tag{2.17}$$

where $$\rho_{\tilde{v}}\langle s', s''\rangle = E[\tilde{v}\langle s'\rangle\, \tilde{v}\langle s''\rangle],$$

and is the 'Lagrangian' covariance of the turbulent velocity component $\tilde{v}$ of a fluid particle at two points distances s' and s'' along the mean streamline from the source. These points may or may not be distinct and may or may not be on the mean streamline. Since $\rho_{\tilde{v}}\langle s', s''\rangle$ is impossible to calculate and difficult to measure, it is more useful to express $\overline{N^2}(s)$ in terms of a covariance which can be measured by standard techniques. From (2.15*b*) and (2.16*b*) it follows that in the limit $\alpha \to 0$, $\delta \to 0$, $(\alpha a/L_E) \to 0$

$$\overline{N^2} = \frac{1}{\overline{U}_0^2(s)}\int_0^s\int_0^s \rho_{\tilde{v}_0}\langle s', s''\rangle\, ds'\, ds'', \tag{2.18}$$

where $$\rho_{\tilde{v}_0}\langle s', s''\rangle = E[\tilde{v}_0\langle s'\rangle\, \tilde{v}_0\langle s''\rangle]. \tag{2.19}$$

But since $\tilde{v}(s, n, z, t)$ is a stationary random function of time

$$\rho_{\tilde{v}_0}\langle s', s''\rangle = \rho_{\tilde{v}_0}(s', s'', T_0(s'') - T_0(s')), \tag{2.20}$$

where, writing out $\tilde{v}_0(s, t)$ in full,

$$\rho_{\tilde{v}_0}(s', s'', T_0(s'') - T_0(s')) = \overline{\tilde{v}_0(s', 0, 0, t)\, \tilde{v}_0(s'', 0, 0, T_0(s'') - T_0(s') + t_0)},$$

$T_0(s)$ being given by (2.7). Thus the mean-square dispersion is approximately given by a double integral of the covariance function $\rho_{\tilde{v}_0}\langle s', s''\rangle$. This may be described as a 'pseudo-Lagrangian'† covariance because to first order it is the covariance at two points of the turbulent velocity component $\tilde{v}$ coincident with a *fluid particle* at those two *points*. It is not an exact Lagrangian covariance because the turbulent velocity is measured on the mean streamline, i.e. at $(s', 0)$ and $(s'', 0)$, not at the displacement positions of the particle, $(s', N(s'))$ and $(s'', N(s''))$, hence the prefix 'pseudo-'. Note that $\rho_{\tilde{v}_0}(s', s''; T_0(s'') - T_0(s'))$ given by (2.20) is the cross-covariance of $\tilde{v}$ at two points s' and s'' along the mean streamline, but with a time delay equal to the average time it takes a fluid particle to travel from $s = s'$ to $s' = s''$ along the streamline.

Perhaps the most surprising features of the results (2.17) and (2.18) are first that $\overline{N^2}$ is calculated from the covariance of the velocity of a fluid particle at separate points in *space*, not time, as is usual in turbulent dispersion calculations, and second that the value of the mean velocity is only directly involved at the

† In the terminology of Corrsin and Lumley (Snyder & Lumley 1971), this would be described as an Eulerian correlation in moving co-ordinates.

point at which $\overline{N^2}(s)$ is required. Of course, $\tilde{U}_0(s)$ is needed implicitly to calculate $T_0(s'') - T_0(s')$.

Given the conditions that $\alpha \ll 1$ and $\alpha a/L_E \ll 1$, which are necessary for the validity of (2.18), Hunt (1973) has shown that $\rho_{\tilde{v}_0}\langle s', s''\rangle$ can be calculated in terms of the Eulerian energy spectrum tensor of the upstream turbulence by using a generalization of the methods of rapid-distortion theory (Batchelor & Proudman 1954). An example is given in § 2.2 of the use of this theory to calculate the dispersion in a particular case.

It is now possible to justify the assumption (2.6) in terms of (2.5). Analysis of (2.17) and (2.18) shows that, for all scales of turbulence, if $s = O(a)$ as $\alpha \to 0$, $\delta \to 0$

$$\overline{N^2} = O(\alpha^2 a^2). \tag{2.21}$$

Therefore, as defined in (2.6),

$$\delta = O(\alpha), \tag{2.22}$$

which confirms the assumption of (2.6) that, if $\alpha \ll 1$, $\delta \ll 1$. Thus the ratio $\alpha a/L_E = O(\delta a/L_E)$, so that the limitation on the solution (2.18) physically implies that $L_E \gg \delta$. For if $L_E < a\delta$ it is clear that $\tilde{v}_0(s,t)$ could not be used as an approximation for $\tilde{v}(s, N, Z, t)$.

If a/L_E is $O(1)$ it is possible to calculate N more accurately, by expanding N and Δ_2 as series in α:

$$N = \alpha N^{(0)} + \alpha^2 N^{(1)} + \ldots, \quad \Delta_2 = U_\infty(\alpha^2 \Delta_2^{(0)} + \alpha^3 \Delta_2^{(1)} + \ldots), \tag{2.23}$$

where $\alpha N^{(0)}$ is given by (2.15*b*) and $N^{(1)}$ is given by the solution of

$$d(U_0\langle s\rangle N^{(1)}\langle s\rangle)/ds = U_\infty \Delta_2^{(0)}\langle s\rangle, \tag{2.24}$$

where $\Delta_2^{(0)}\langle s\rangle$ is calculated from (2.14) using $N^{(0)}\langle s\rangle$. Thence the error in $\overline{N^2}$ can be calculated and will be $O(\alpha^3 a^2)$.

More interesting than the error in $\overline{N^2}$ is the consequence of (2.24) that, although $\overline{N^{(0)}}(s) = 0$,

$$E[\alpha^2 N^{(1)}\langle s\rangle] = \alpha^2 \overline{N^{(1)}}(s) \neq 0,$$

because $E[\Delta^{(0)}\langle s\rangle]$ contains terms like

$$E[N^2]\frac{\partial^2 \tilde{V}}{\partial n^2}\langle s\rangle, \quad \frac{1}{\tilde{U}_0\langle s\rangle}\int_0^s E\left[v\langle s'\rangle \frac{\partial v}{\partial n}\langle s'\rangle\right] ds'.$$

Thus to *second* order the mean path of fluid particles, which is the centre of the plume, does *not* follow the mean streamline. As the terms in the expression for $E[\Delta^{(0)}\langle s\rangle]$ show, this divergence is caused by (*a*) variations normal to the mean streamline of the mean velocity components $\tilde{U}$ and $\tilde{V}$ and their derivatives, and of the turbulent velocity $\tilde{v}$ and (*b*) the Reynolds stress on the mean streamline $\overline{\tilde{u}\tilde{v}}$. The latter effect has been known theoretically and observed experimentally since the work of Hinze & van der Hegge Zijnen (1951)†, but the former effect has not been noticed before.

Now consider the displacement $Z(s,t)$ in the plane perpendicular to the flow (the z direction of figure 1) of a fluid particle emanating from the source. If $\tilde{w}$

† See also the paper by Batchelor (1964).

is the turbulent velocity in the z direction then, since there is no mean velocity in the z direction,

$$dZ/dT = \tilde{w}\langle T\rangle.$$

Since $Z = 0$ when $T = 0$,

$$Z\langle T\rangle = \int_0^T \tilde{w}\langle T'\rangle\, dT' \tag{2.25}$$

and

$$\overline{Z^2}\langle T\rangle = \int_0^T \int_0^T \rho_{\tilde{w}}\langle T', T''\rangle\, dT'\, dT'', \tag{2.26}$$

where

$$\rho_{\tilde{w}}\langle T'\, T''\rangle = E\,[\tilde{w}\langle T'\rangle\, \tilde{w}\langle T''\rangle].$$

$\rho_{\tilde{w}}\langle T'T''\rangle$ is the usual Lagrangian covariance introduced by Taylor (1921) of the velocity component $\tilde{w}$ of a fluid particle at two moments in time, T' and T''; i.e. at two points $(s(T'), N(T'), Z(T'))$ and $(s(T''), N(T''), Z(T''))$.

Finally it may be of interest to calculate the dispersion of a cloud of matter emitted from the source, in which case we must calculate the displacement of a particle *along* the streamline, $S\langle T\rangle$, which must satisfy the equation

$$dS/dT = \bar{U}\langle T\rangle + \tilde{u}\langle T\rangle = \bar{U}_0\langle T\rangle + N\langle T\rangle\,(\partial\bar{U}/\partial n)_{n=0}\langle T\rangle + \tilde{u}\langle T\rangle.$$

But the mean displacement $\bar{S}$ satisfies

$$d\bar{S}/dT = \bar{U}_0$$

and therefore

$$d(S-\bar{S})/dT = \tilde{u}\langle T\rangle + N\langle T\rangle\,(\partial\bar{U}/\partial n)_{n=0}. \tag{2.27}$$

Thence, from (2.15b),

$$(S-\bar{S})\langle T\rangle = \int_0^T \tilde{u}\langle T'\rangle\, dT' + \int_0^T \left\{\frac{1}{\bar{U}_0}\left(\frac{\partial\bar{U}}{\partial n}\right)_{n=0} \left[\int_0^{T'} (\tilde{v}\langle T''\rangle/\bar{U}_0\langle T''\rangle)\, dT''\right] dT'\right\},$$

whence $\overline{(S-\bar{S})^2}$ can be calculated. But on the stagnation line of a body, where $\partial\bar{U}/\partial n = 0$,

$$\overline{(S-\bar{S})^2} = \overline{S^2} - \bar{S}^2 = \int_0^T \int_0^T \rho_{\tilde{u}}\langle T', T''\rangle\, dT'\, dT'', \tag{2.28}$$

where

$$\rho_{\tilde{u}}\langle T', T''\rangle = \langle \tilde{u}(T')\,\tilde{u}(T'')\rangle.$$

On the other hand, when $\partial\bar{U}/\partial n$ is significant, the second term becomes dominant. In the case of a uniform shear flow where $\partial\bar{U}/\partial n$ is a constant, we recover Corsin's result quoted by Tennekes & Lumley (1972, p. 232), that as $T \to \infty$

$$\overline{(S-\bar{S})^2} \sim \frac{2}{3}\left(\frac{\partial\bar{U}}{\partial n}\right)^2 T^3 \int_0^\infty \rho_{\tilde{v}}\langle T', T'+\tau\rangle\, d\tau. \tag{2.29}$$

Since in a uniform shear flow, as $T \to \infty$,

$$\overline{N^2} \sim 2T \int_0^\infty \rho_{\tilde{v}}\langle T', T'+\tau\rangle\, d\tau,$$

in a uniform shear flow, a cloud becomes elongated in the flow direction. The results for $\overline{(S-\bar{S})^2}$ on the stagnation line in § 2.2 will show how these effects are modified for flow round a body.

In many diffusion problems by inspecting the relevant integrals (2.15), (2.26) or (2.27) it is possible to draw some general conclusions about the dispersion far from the source. This is not possible in this problem until we have studied how the flow round the body changes $\rho_{\tilde{v}}$, $\rho_{\tilde{w}}$ and $\rho_{\tilde{u}}$. However, near the source we can follow Taylor in showing that

$$(\overline{N^2})^{\frac{1}{2}}, (\overline{Z^2})^{\frac{1}{2}}, (\overline{S^2}-\bar{S}^2)^{\frac{1}{2}} = (s/\tilde{U})\{(\overline{\tilde{v}^2})^{\frac{1}{2}}, (\overline{w^2}), (\overline{\tilde{u}^2})^{\frac{1}{2}}\} \quad (s=0). \tag{2.30}$$

That is, the r.m.s. dispersion is proportional to distance from the source.

The error introduced by neglecting the effect of molecular diffusivity can be estimated. Saffman (1960) has shown that, for isotropic turbulence in a uniform flow when $\overline{\omega^2}T^2 \ll 1$,

$$\overline{N^2} = \overline{N_0^2} + 2DT - \tfrac{1}{9}DT\overline{\omega^2}T^2 + \text{higher-order terms}, \tag{2.31}$$

where $\overline{N_0^2}$ is $\overline{N^2}$ calculated by Taylor's statistical argument neglecting the molecular diffusivity D, and $\overline{\omega^2}$ is the mean-square turbulent vorticity. The third term in this expansion represents the interaction between molecular and turbulent diffusive actions. For non-homogeneous flows one would still expect the magnitude of molecular diffusion effects to be similar to those of the second and third terms in (2.31). When $\alpha \ll 1$ and $\alpha a/L_E \ll 1$ we have shown in (2.21) that $\overline{N_0^2} \sim \overline{u_{\infty 2}^2}a^2/U_\infty^2$. Hence the error in neglecting molecular diffusivity is of order $(2/\alpha)\,[D/(a(\overline{u_\infty^2})^{\frac{1}{2}})]$, which is much less than 1 for all but the weakest turbulence.

When $\overline{\omega^2}T^2 \gg 1$ Saffman has estimated the third term in (2.31) to be of order $D\overline{N_0^2}/(\nu R_\lambda^{\frac{1}{2}})$, where R_λ is the Reynolds number based on the dissipation length λ. For many laboratory-scale experiments $\overline{\omega^2}T^2 \ll 1$, but for full-scale experiments where $\overline{\omega^2}T^2 \gg 1$, taking a wind speed of 10 m/s and $\lambda \sim 0{\cdot}3$ cm, $R_\lambda \sim 2000$. Hence in the full-scale situation molecular diffusivity will still have a small effect, given that the source is only a few diameters upstream of the body. See § 4 for further comments on this limitation.

An important limitation to our general results (2.15), (2.26) or (2.27) is that they are not valid when $(\overline{N^2})^{\frac{1}{2}}$ is greater than the distance from the source streamline to the body. On the body, $\tilde{v} = 0$, yet $\tilde{v} \neq 0$ on the source streamline, so that if the effect of the body is to be analysed we shall have to allow for the effect of the mean statistical properties of $\tilde{v}$ varying across the plane. This is a problem yet to be studied.

2.2. *Diffusion in a turbulent flow round a circular cylinder*

We shall assume that $\alpha \ll 1$, $\alpha a/L_E \ll 1$ and in addition that $L_E \gg a$. In that case the turbulent velocity distribution can be calculated using a 'quasi-steady' assumption (Hunt 1973). The turbulence induced by the wake is not known and will be neglected, although if known it could easily be incorporated into the results.

If u_i denotes the x, y or z component of velocity near the body, then for large-scale turbulence u_i can be expressed in terms of the jth component of velocity upstream by the matrix $M_{ij}(x,y)$: $u_i(x,y,z,t) = M_{ij}(x,y)\, u_{\infty j}(z,t)$. Assume x and y to be normalized in terms of the cylinder's radius a, and concentrating on the flow forward of the separation point, we assume that the mean (and hence the turbulent) velocities are approximately given by potential-flow theory, so that

$$M_{ij} = \begin{bmatrix} 1-(x^2-y^2)/(x^2+y^2)^2 & -2xy/(x^2+y^2)^2 & 0 \\ -2xy/(x^2+y^2)^2 & 1+(x^2-y^2)/(x^2+y^2)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.32}$$

Thence the turbulent velocity normal to the streamlines

$$\tilde{v} = (u_2 - m u_1)/(1+m^2)^{\frac{1}{2}}, \tag{2.33}$$

where

$$m = -2xy/[(x^2+y^2)^2-(x^2-y^2)] \tag{2.34a}$$

and is the slope of the streamline.

From (2.32) and (2.33) it follows that

$$M_{2j} - m M_{1j} = 0,$$

whence

$$\tilde{v}(x,y,z,t) = u_{\infty 2}(t)\, F(x,y), \tag{2.34b}$$

where

$$F(x,y) = (M_{22} - m M_{21})/(1+m^2)^{\frac{1}{2}}$$

$$= \left[1 + \frac{x^2-y^2}{(x^2+y^2)^2} + \frac{2mxy}{(x^2+y^2)^2}\right] \Big/ (1+m^2)^{\frac{1}{2}}. \tag{2.35}$$

Thus the x component $u_{\infty 1}$ of turbulent velocity upstream, does not provide any contribution to $\tilde{v}$. If the source at (x_p, y_p) lies on the streamline with stream function ψ, where $\psi = y_p[1-(1/(x_p^2+y_p^2))]$, then at any x the y co-ordinate of the streamline can be calculated from the equation

$$\psi = y[1-(1/(x^2+y^2))]. \tag{2.36}$$

Thus, for a given source,

$$F(x,y) = F(x,\psi),$$

whence $\rho_{\tilde{v}_0}\langle s', s''\rangle$ becomes

$$\rho_{\tilde{v}_0}\langle s', s''\rangle = \overline{u_{\infty 2}(T')\, u_{\infty 2}(T'')}, \tag{2.37}$$

where the time taken to travel between s' and s'', or x' and x'', is $T''-T'$. Note that $\rho_{\tilde{v}_0}$ has been transformed to a function of an *Eulerian* autocorrelation. We can transform (2.37) again because

$$\overline{u_{\infty 2}(T')\, u_{\infty 2}(T'')} = \overline{u^2_{\infty 2}}\, R_{22}(T''-T'),$$

where $R_{22}(T''-T')$ is the Eulerian autocorrelation in the uniform flow upstream.

But since $T''-T' = O(a/U_\infty)$ and since

$$a/U_\infty \ll \int_0^\infty R_{22}(\tau)\, d\tau = O(L_E/U_\infty),$$

it follows that $R_{22}(T''-T') = 1$. Therefore $\rho_{\tilde{v}_0}\langle s',s''\rangle = \overline{u^2_{\infty 2}}F(x',\psi)F(x'',\psi)$, whence

$$(\overline{N^2})^{\frac{1}{2}} = \frac{(\overline{u^2_{\infty 2}})^{\frac{1}{2}}}{\tilde{U}_0(x,\psi)}\int_{x_p}^{x'}[1+m^2(x')]^{\frac{1}{2}}F(x',\psi)\,dx'. \tag{2.38}$$

This integral has been computed for a number of values of y_p when $x_p = -3$ and the results are shown in figure 2(*a*) for a particular value of $(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty$ and in figure 2(*b*) for general value of $(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty$. Rapid growth of $(\overline{N^2})^{\frac{1}{2}}$ can be seen on the stagnation streamline, while for sources off the stagnation streamline it can be seen that in the x, y plane the plumes first expand and then contract as the streamlines converge at the sides of the body. For large-scale and weak turbulence this phenomenon occurs whatever the value of $(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty$. Note that it follows from (2.15) and (2.34*b*) that if $u_{\infty 2}(t)$ has a Gaussian probability distribution so has N at a particular value of s and the concentration profile of the plume is then also Gaussian.

Since in the limit

$$L_E \gg a,$$

$$\tilde{w}(x,y,z,t) = u_{\infty 3}(z,t),$$

from (2.26)

$$(\overline{Z^2})^{\frac{1}{2}} = (\overline{u^2_{\infty 3}})^{\frac{1}{2}}\int_{x_p}^{x} dx'/U(x',\psi), \tag{2.39}$$

which shows that, unlike $(\overline{N^2})^{\frac{1}{2}}$, the plume always grows in the z direction downstream from the source, wherever the source may be situated. This point is illustrated by the difference in the graphs of $(\overline{N^2}(x))^{\frac{1}{2}}$ and $(\overline{Z^2}(x))^{\frac{1}{2}}$ for the plume originating at $(-3{\cdot}12, 0{\cdot}5)$, shown in figures 2(*b*) and (*c*).

Since $\tilde{u} = (u_1+mu_2)/(1+m^2)^{\frac{1}{2}}$, from (2.32) it follows that

$$\tilde{u}(x,y,z,t) = u_{\infty 1}(z,t)\,G_1(x,\psi)+u_{\infty 2}(z,t)\,G_2(x,\psi),$$

where

$$G_1(x,\psi) = \left[1-\frac{x^2-y^2+2xym}{(x^2+y^2)^2}\right]\Big/(1+m^2)^{\frac{1}{2}},$$

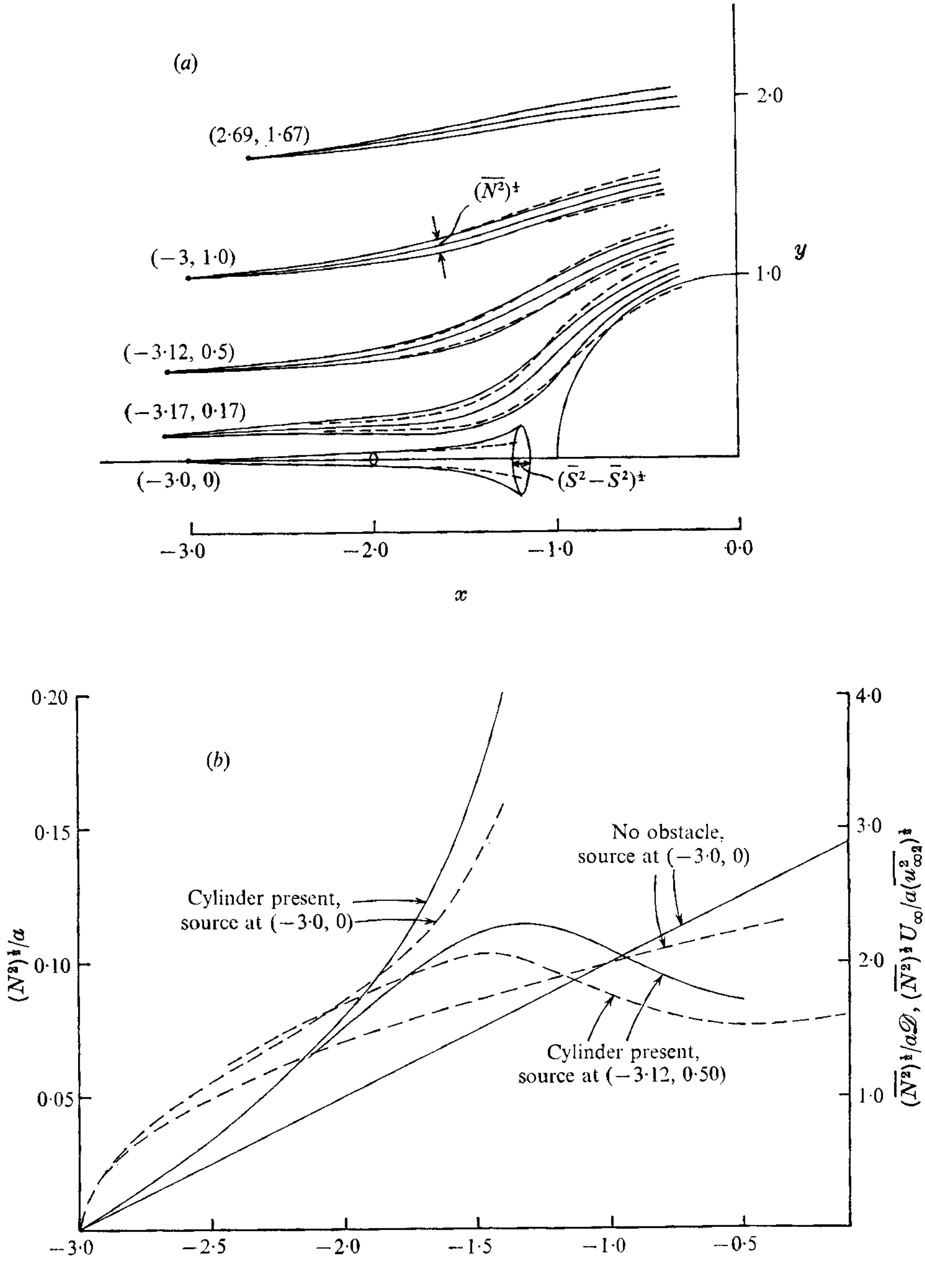
$$G_2(x,\psi) = \left[m+\frac{(x^2-y^2)\,m-2xy}{(x^2+y^2)^2}\right]\Big/(1+m^2)^{\frac{1}{2}},$$

and $y = y(x,\psi)$ is the solution of (2.36). Following arguments similar to those for $(\overline{N^2})^{\frac{1}{2}}$, but making the additional assumption of isotropy of the incident turbulence so that $\overline{u^2_{\infty 1}} = \overline{u^2_{\infty 2}}$ and $\overline{u_{\infty 1}u_{\infty 2}} = 0$, equation (2.28) leads to the following result for a source on the stagnation line:

$$(\overline{S^2}-\overline{S}^2)^{\frac{1}{2}} = (\overline{u^2_{\infty 1}})^{\frac{1}{2}}\left\{\left[\int_{x_0}^{x_1}\frac{G_1(x',\psi)\,dx'}{U(x',\psi)}\right]^2+\left[\int_{x_0}^{x}\frac{G_2(x',\psi)\,dx'}{U(x',\psi)}\right]^2\right\}. \tag{2.40}$$

This result is also shown graphically in figure 2.

Consider the particular case of the dispersion from a source placed on the



FIGURES 2 (a), (b). For legend see facing page.

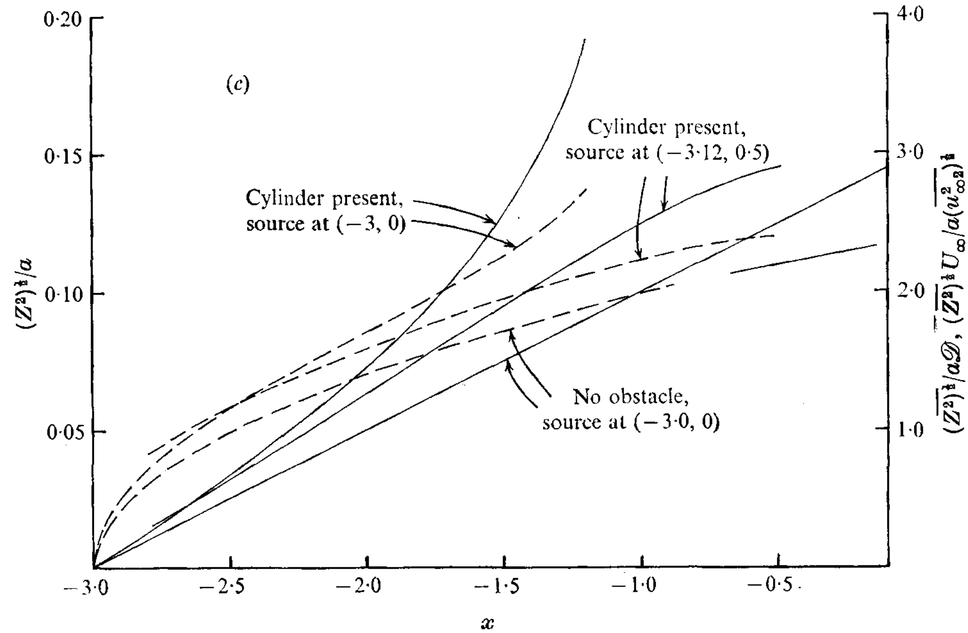


FIGURE 2. Dispersion from sources near a circular cylinder. (*a*) Plume widths for large-scale turbulence with $(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty = 0{\cdot}05$. Source positions are indicated. The central curve from each source position is the streamline on which it is situated. The distance between the two full lines on either side is $(\overline{N^2})^{\frac{1}{2}}$ and is drawn to scale. The distance between the dashed lines is $(\overline{Z^2})^{\frac{1}{2}}$; the width of the ellipses in the streamwise direction is $(\overline{S^2}-\overline{S}^2)^{\frac{1}{2}}$. (*b*), (*c*) Comparison of values of $(\overline{N^2})^{\frac{1}{2}}$ and $(\overline{Z^2})^{\frac{1}{2}}$ for sources on and off the stagnation line, calculated by the statistical (solid line, $L_E \gg a$) and diffusion-equation (broken line) methods. Left-hand axes are scaled for $(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty = 0{\cdot}05$ and $(\overline{u^2_{3\infty}})^{\frac{1}{2}}/U_\infty = 0{\cdot}05$. Right-hand axes are scaled for arbitrary values of $(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty$ and $(\overline{u^2_{\infty 3}})^{\frac{1}{2}}/U_\infty$ and $\mathscr{D}$.

stagnation line, the only one for which simple exact solutions are possible. If $x_p = -X_p$, then

$$(\overline{N^2}(x))^{\frac{1}{2}} = \frac{(\overline{u^2_{\infty 2}})^{\frac{1}{2}}}{U_\infty(1-1/x^2)}\int_{-X_p}^{x}\left(1+\frac{1}{(x')^2}\right)dx'$$

$$= [(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty]\,[x-(1/x)+X_p-1/X_p]/(1-1/x^2). \qquad (2.41)$$

Thence as $x \to -1$,

$$(\overline{N^2})^{\frac{1}{2}} \sim -[(\overline{u^2_{\infty 2}})^{\frac{1}{2}}/U_\infty]\,(X_p-1/X_p)/[2(1+x)],$$

so that the width of the plume apparently becomes infinite! In fact when

$$-(1+x)\,U_\infty/(\overline{u^2_{\infty 2}})^{\frac{1}{2}} \ll 1$$

the assumption (2.6) that $(\overline{N^2})^{\frac{1}{2}} \ll a$ is violated so the analysis is invalid. The value for $(\overline{N^2})^{\frac{1}{2}}$ around the cylinder is then also wrong. Figures 2 (*a*) and (*b*) clearly show the broadening of the plume in this region:

$$(\overline{Z^2})^{\frac{1}{2}} = \frac{(\overline{u^2_{\infty 3}})^{\frac{1}{2}}}{U_\infty}\int_{-X_p}^{x}\frac{dx'}{1-(1/x')^2}$$

$$= \frac{(\overline{u^2_{\infty 3}})^{\frac{1}{2}}}{U_\infty}\left\{x+X_p+\tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\,\frac{X_p-1}{X_p+1}\right|\right\}, \qquad (2.42)$$

whence as $x \rightarrow -1$,

$$(\overline{Z^2})^{\frac{1}{2}} \sim [(\overline{u^2_{\infty 3}})^{\frac{1}{2}}/U_\infty]\{-\tfrac{1}{2}\ln|x+1|\}.$$

Thus as the stagnation point is approached the plume widens rapidly in the z direction but not as rapidly in the y direction, as shown by figures 2(a) and (c).

Finally, if the turbulence is isotropic,

$$\{\overline{S^2} - \bar{S}^2\}^{\frac{1}{2}} = [(\overline{u^2_{\infty 1}})^{\frac{1}{2}}/U_\infty](x+X_p). \tag{2.43}$$

A cloud only widens linearly along the stagnation line, and therefore becomes like a flattened rugby football! Comparing this effect with that due to shear indicates that, for sources a little off the stagnation line where $\partial\tilde{U}/\partial n \neq 0$, the two effects compete with each other, perhaps leading to no significant change in cloud shape from spherical.

3. Diffusion-equation methods

3.1. *Justification of the method*

The statistical analysis has provided a method for calculating the plume width. By assuming a probability density distribution for $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$, the method can be extended to predict the distribution for the concentration C. However, so far we have not been able to extend the method to find $C(x,y)$ close to the body, and therefore we have to consider using the diffusion equation. Scriven (1970) has proved that the diffusion equation can be used satisfactorily to predict maximum ground-level concentrations, or at least upper and lower limits. But, as Pasquill (1971) has pointed out, the distribution of C along the ground is not well predicted.

However, we have seen in § 2 that in a straining flow the converging or diverging of streamlines may be as important as the diffusion process. Therefore the correct modelling of the diffusion terms may be less important in demonstrating the general effects on the distribution of C of flow over buildings and hills, a subject about which little is known but more ought to be (Pasquill 1972).

3.2. *Diffusion from a line source*

The two-dimensional diffusion equation with a constant coefficient may be written as

$$U'\frac{\partial C}{\partial x} + V'\frac{\partial C}{\partial y} = \mathscr{D}\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right), \tag{3.1}$$

where U' and V' are the x and y components of mean velocity normalized with respect to U_∞. x and y are normalized with respect to a dimension a of the body (the radius in the case of a cylinder), and

$$C = C^* a U_\infty / Q, \tag{3.2}$$

C† being the time-mean concentration at (x,y) and Q the quantity of matter

† C is the quantity K_c described by Halitsky (1968) as a concentration coefficient.

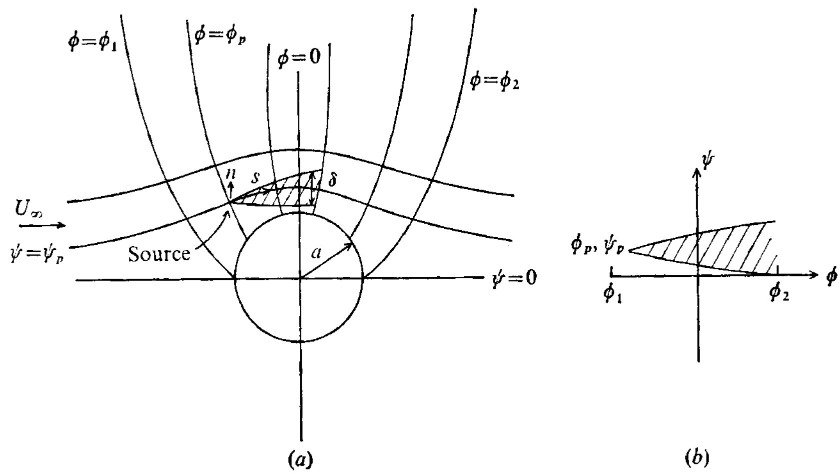


FIGURE 3. (*a*) Plume in x, y space. (*b*) Plume in ϕ, ψ space.

emitted by a *line* source per unit time per unit width. Finally, the normalized diffusion coefficient

$$\mathscr{D} = (D+K)/(aU_\infty), \tag{3.3}$$

where D is the molecular diffusivity and where the appropriate K is the eddy diffusivity, which is, unless otherwise stated, assumed to be constant.

If we now assume that the flow round the body is irrotational then the general solution to (3.1) for a line source is that found by King (1914) by using Boussinesq's (1905) transformation of (3.1). If we write

$$U' = \partial\phi/\partial x = \partial\psi/\partial y,$$

$$V' = \partial\phi/\partial y = -\partial\psi/\partial x,$$

(3.1) becomes

$$\frac{\partial C}{\partial \phi} = \mathscr{D}\left(\frac{\partial^2 C}{\partial \phi^2}+\frac{\partial^2 C}{\partial \psi^2}\right). \tag{3.4}$$

Then the solution for a line source placed at $\phi = \phi_p$, $\psi = \psi_p$ (see figure 3*a*) is

$$C = \frac{1}{2\pi\mathscr{D}}\exp\left(\frac{\phi-\phi_p}{2\mathscr{D}}\right)K_0\left(\frac{[(\phi-\phi_p)^2+(\psi-\psi_p)^2]^{\frac{1}{2}}}{2\mathscr{D}}\right), \tag{3.5}$$

where $K_0(x)$ is a modified Bessel function. In this solution the boundary conditions on the body at $\psi = 0$ have been ignored, so that the solution is valid if the width of the plume is small compared with the distance of the streamline $\psi = \psi_p$ from the body, i.e.

$$\delta/a \ll \psi_0. \tag{3.6}$$

If (3.6) is satisfied, and if

$$\mathscr{D} \ll 1, \tag{3.7}$$

then using the asymptotic form for $K_0(x)$, (3.5) becomes

$$C = \frac{1}{[4\pi\mathscr{D}(\phi-\phi_p)]^{\frac{1}{2}}}\exp\{-(\psi-\psi_p)^2/(4\mathscr{D}(\phi-\phi_p))\}. \tag{3.8}$$

This is found to be a good approximation in many cases of practical interest. If the further approximation is made that the plume is sufficiently thin to assume that, if $\tilde{U}'$, s and n are the *non-dimensional* forms of $\tilde{U}$, s and n (defined in figure 1),

$$\psi - \psi_p \simeq n\tilde{U}',$$

which is valid everywhere *except* near the stagnation point, then (3.8) becomes

$$C = \frac{1}{\left(4\pi\mathscr{D}\int_0^s q(s')\,ds'\right)^{\frac{1}{2}}} \exp\left\{-n^2\tilde{U}'^2 \Big/ \left[4\mathscr{D}\int_0^s \tilde{U}'(s')\,ds'\right]\right\}. \tag{3.9}$$

3.3. *Effects of the boundary*

The body not only affects the dispersion by affecting the velocity, but also by imposing a boundary condition on the concentration. If the body is assumed to be impermeable to the pollutant, then the boundary condition is

$$\partial C/\partial n = \partial C/\partial \psi = 0.\dagger \tag{3.10}$$

For sources on the stagnation streamline the solution (3.5) satisfies (3.10) identically, but to satisfy (3.10) for sources off the stagnation streamline is more difficult. Formal solutions have been obtained as infinite series of parabolic cylinder functions as shown by Gill (1960), but these have not yet been evaluated.

3.4. *Asymptotic solutions for diffusion from point and line sources*

If we consider a *point* source on a streamline in any two-dimensional rotational or irrotational flow and if we assume $\mathscr{D}$ to have different values for diffusion in the two directions normal to the mean streamline then, in the limit $\mathscr{D} \to 0$, the equation governing convective diffusion in terms of the non-dimensional streamline co-ordinates (s, n, z) is

$$\tilde{U}'\frac{\partial C}{\partial s} + n\left(\frac{\partial \tilde{V}'}{\partial n}\right)_{n=0}\frac{\partial C}{\partial n} = \mathscr{D}_1(s)\frac{\partial^2 C}{\partial n^2} + \mathscr{D}_2(s)\frac{\partial^2 C}{\partial z^2}. \tag{3.11}$$

We assume that $\mathscr{D}_1$ and $\mathscr{D}_2$ may be functions of the distance s along the streamlines but do not vary across the plume. This assumption is exactly similar to that we used for the analysis of turbulent diffusion in § 2. By continuity, the equation becomes

$$\tilde{U}'_0\frac{\partial C}{\partial s} - n\frac{\partial \tilde{U}'_0}{\partial s}\frac{\partial C}{\partial n} = \mathscr{D}_1(s)\frac{\partial^2 C}{\partial n^2} + \mathscr{D}_2(s)\frac{\partial^2 C}{\partial z^2}, \tag{3.12}$$

† In general the boundary layer on a body cannot be ignored when calculating mass or heat transfer to its surface. But if the body is impermeable, the external flow is turbulent and the source is outside the boundary layer, then the boundary layer of the body is probably unimportant.

to which the solution for a *point* source of strength Q units of matter per unit time is

$$C(s,n,z)=\frac{\exp\left\{-\left[n^2\tilde{U}'^2\Big/\left(4\int_0^s \mathscr{D}_1(s')\,\tilde{U}_0'\,ds'\right)+z^2\Big/\left(4\int_0^s \mathscr{D}_2(s')\frac{ds'}{\tilde{U}_0(s')}\right)\right]\right\}}{4\pi\left\{\int_0^s \mathscr{D}_1(s')\,\tilde{U}_0(s')\,ds'\int_0^s \mathscr{D}_2(s')\,ds'/\tilde{U}_0(s')\right\}^{\frac{1}{2}}}, \quad (3.13)$$

where $C = C^* a^2 U_\infty/Q$. We are not aware of a previous derivation of this solution, although we are aware of the attempts by Stumke (1964) to find approximate solutions to (3.11), with and without a variable diffusion coefficient. The solution for a *line* source can be obtained by integrating with respect to z:

$$C(s,n)=\frac{\exp\left\{-n^2\tilde{U}'^2\Big/\left(4\int_0^s \mathscr{D}_1(s')\,\tilde{U}_0(s')\,ds'\right)\right\}}{\left\{4\pi\int_0^s D_1(s')\,\tilde{U}_0(s')\,ds'\right\}^{\frac{1}{2}}}, \quad (3.14)$$

where C is defined by (3.2). This solution was first obtained by Burger (1964), but he did not appreciate that it is equally valid for a rotational as for an irrotational flow: thus, for example, it is valid for diffusion from a line source in a shear flow. Note that, if $\mathscr{D}$ is constant, (3.13) is identical to (3.9).

To allow for the effects of boundaries additional terms must be constructed. To compare the solutions (3.9), (3.13) and (3.14) with the values of $\overline{N^2}$ and $\overline{Z^2}$ calculated by statistical methods in §2, we have to define $\overline{N^2}$ and $\overline{Z^2}$ in terms of C in the same way in both cases. Batchelor & Townsend (1956) show that for turbulent diffusion

$$\overline{N^2}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n^2C^*\,dn\,dz\Big/\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C^*\,dn\,dz, \quad (3.15)$$

$$\overline{Z^2}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z^2C^*\,dn\,dz\Big/\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C^*\,dn\,dz. \quad (3.16)$$

Thence from (3.12)

$$\overline{N^2}=\frac{2a^2}{\tilde{U}_0'^2(s)}\int_0^s \mathscr{D}(s')\,\tilde{U}_0'(s')\,ds'$$

$$=\frac{2}{\tilde{U}^2(s)}\int_0^s \mathscr{D}(s')\,\tilde{U}_0(s')\,ds' \quad (3.17)$$

and

$$\overline{Z^2}=2a^2\int_0^s\frac{\mathscr{D}(s')\,ds'}{\tilde{U}'(s')}=2\int_0^s\frac{\mathscr{D}(s')\,ds'}{\tilde{U}(s')}. \quad (3.18)$$

3.5. *Results for sources near a circular cylinder*

Assuming a constant diffusivity and using (3.8) and (3.17), $(\overline{N^2})^{\frac{1}{2}}$ has been calculated as a function of s, the distance from the source P, for a number of sources placed in an irrotational flow round a circular cylinder. The results are presented in figures 2(*b*), 2(*c*) and 4, for a value of $\mathscr{D} = 2{\cdot}5\times10^{-3}$, which was

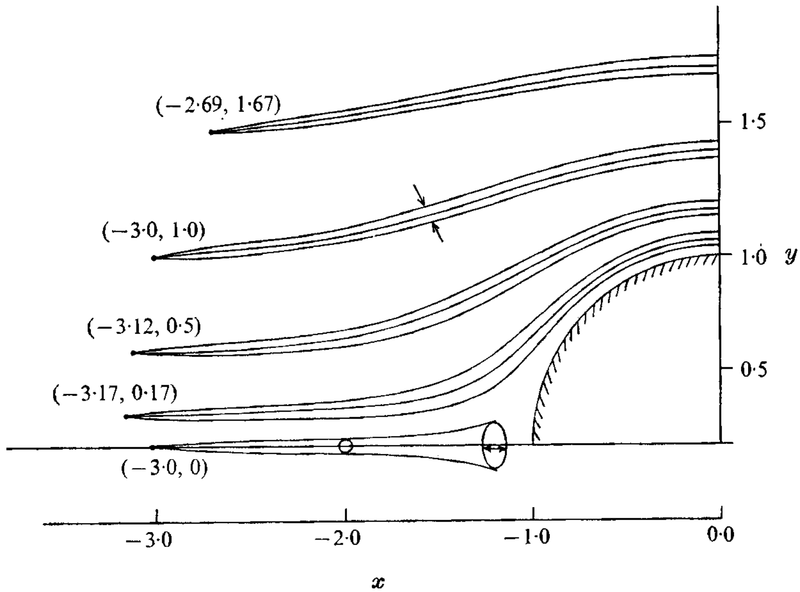


FIGURE 4. Plume widths for constant diffusivity $\mathscr{D} = 2{\cdot}5 \times 10^{-3}$. See caption of figure 2 for meanings of lines.

chosen to give plume widths comparable with those in figure 2(*a*). The solutions (3.5) and (3.13) have been used to calculate the non-dimensional concentration along the centre-line of plumes emitted from line and point sources placed at various upstream positions in the irrotational flow around a circular cylinder. The results are shown in figures 5(*a*) and (*b*), which include the graphs of C against x along the centre-lines of plumes emitted from sources in the absence of an obstacle in the flow.

Now consider the detailed implications of our solutions (3.9) and (3.12).

Source on the stagnation streamline.† Consider first a *line* source at $x_p = -X_p$, $y_p = 0$. Then the solution for C, when $\mathscr{D} \ll 1$ and $n \ll 1$, can easily be found from (3.9):

$$C = \exp(-y^2/\delta^2)/(\pi^{\frac{1}{2}}\delta\tilde{U}'), \tag{3.19}$$

where $\tilde{U}' = 1 - 1/x^2$ and the thickness of the plume δ is given by

$$\left.\begin{aligned}\delta^2 &= (4\mathscr{D}/\tilde{U}'^2)\int_{-X_p}^{x}\tilde{U}'(x')\,dx' \\ &= 4\mathscr{D}[X_p + 1/X_p + x + 1/x]/(1-1/x^2)^2.\end{aligned}\right\} \tag{3.20}$$

Note that for a source on the stagnation streamline $\partial C/\partial n = 0$ on the body so that (3.10) is satisfied. It is particularly interesting to consider C near the

† This might represent dispersion from point sources produced by flows round buildings and hills in very stable atmospheric conditions when all velocities are horizontal and diffusion occurs in horizontal planes.

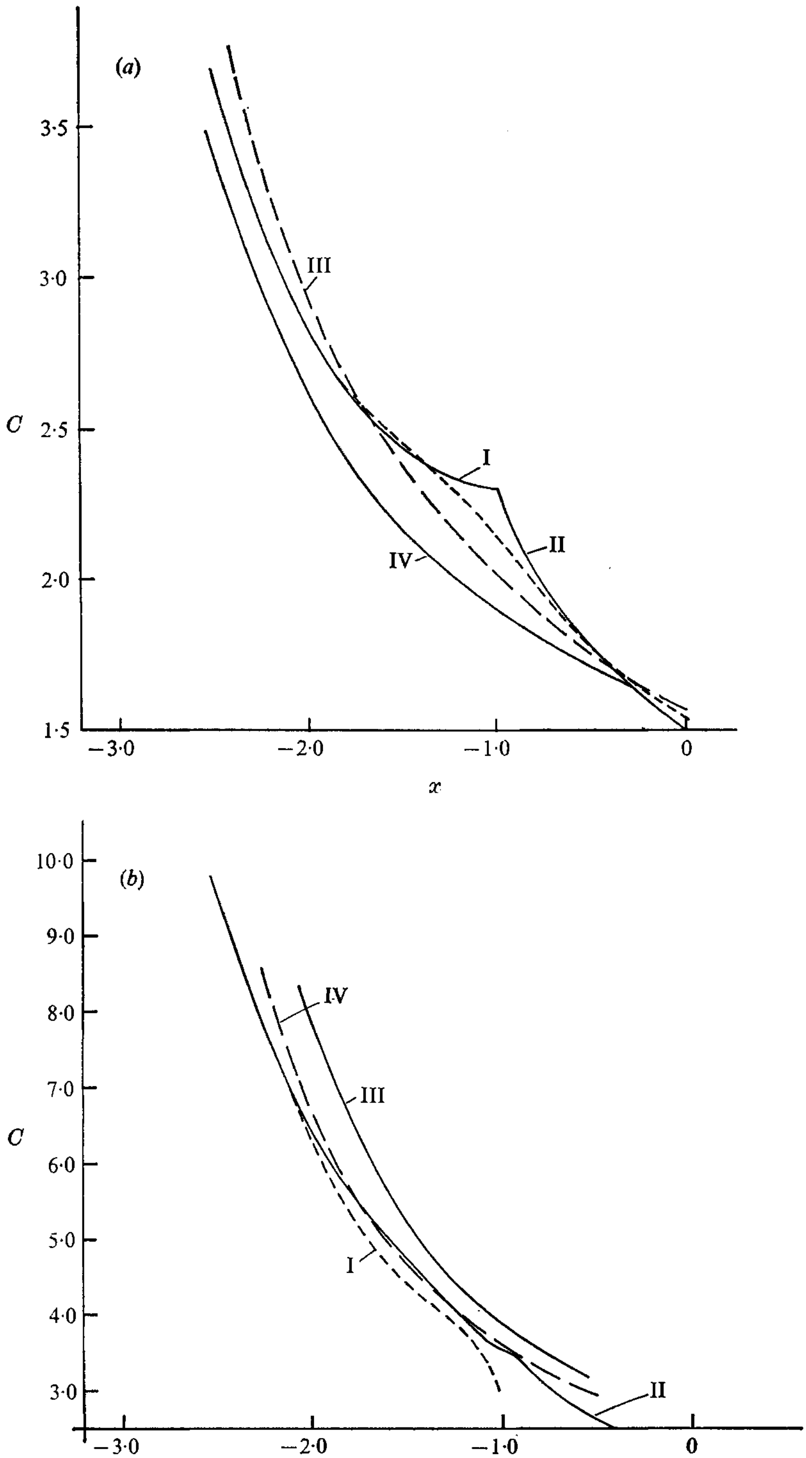


FIGURE 5. Plume centre-line concentrations for (*a*) *line* and (*b*) *point* sources upstream of a circular cylinder; $\mathscr{D} = 10^{-2}$. I, source at (−3·16, 0); II, source at (−3·16, 0·17); III, source at (−3·0, 1·0); IV, source at (−3·16, 0) but no cylinder in the flow.

stagnation point at $x = -1$. Take the half-width of the plume as $(\overline{N^2})^{\frac{1}{2}}$, defined by (3.15), then (3.20) shows that $(\overline{N^2})^{\frac{1}{2}} = \delta/2^{\frac{1}{2}}$. Now as $x \rightarrow -1$,

$$\delta \sim \{\mathscr{D}(X_p + 1/X_p)\}^{\frac{1}{2}}(1+x)^{-1}, \tag{3.21}$$

so that very close to the stagnation point δ becomes singular or, in physical terms, the width of the plume $\delta = 2(\overline{N^2})^{\frac{1}{2}}$ becomes very large. The effect can be seen in figures 2 (*b*) and 4 and is similar to but less marked than that calculated by the statistical analysis of large-scale turbulence as shown in figure 2. The reason why the spreading is less when calculated by the diffusion equation is that near the stagnation point in large-scale turbulence the turbulent velocity component $\tilde{v}$ is nearly doubled, thus amplifying the dispersion caused by the diverging of the streamlines. It is interesting to speculate whether the diffusion equation does in fact approximate to the result of diffusion produced by very small scales of turbulence.

Near the stagnation point $x \rightarrow -1$, the value for the concentration is given by

$$C(y=0) \sim \tfrac{1}{2}\{\pi\mathscr{D}(X_p + 1/X_p - 2)\}^{-\frac{1}{2}}. \tag{3.22}$$

Thus although $(\overline{N^2})^{\frac{1}{2}} \rightarrow \infty$ as $x \rightarrow -1$, *C remains finite*; the reason is that the flux of pollutant remains constant in the plume, i.e.

$$C \times \delta \times \tilde{U}' = \text{constant}. \tag{3.23}$$

Therefore, since, as $x \rightarrow -1$, $\delta \propto |1+x|^{-1}$ and $\tilde{U}^1 \propto |1+x|$, it follows that C remains finite. If we compare the expression for C in (3.22) with the value C would have at $x = -1$ in the absence of the body, C_0, we find

$$C_0(x=-1, y=0) = \tfrac{1}{2}(\pi\mathscr{D}(X_p - 1))^{-\frac{1}{2}}.$$

So that we have the surprising result, which is also visible in figure 5, that $C > C_0$ for all values of X_p; for a typical case when $X_p = 3$, $C = 1{\cdot}2C_0$. The physical reason is that with the mean flow spreading the plume less diffusion occurs normal to the streamlines than for an unobstructed plume.

Calculating $C(x, y)$ using (3.5) and plotting the results in the form of contour lines of constant values of C, shown in figure 6 (*a*), indicates that the greatest change due to the body's presence occurs away from the stagnation point. At $y = 0{\cdot}5$ concentration levels are approximately double, which is simply due to the pollutant being convected round the body. In order to reach the same place, in the absence of the body it would have to be diffused laterally.

Now consider the problem of a *point* source at $(-X_p, 0)$. The solution for C when $\mathscr{D}_1 = \mathscr{D}_2 = \mathscr{D} \ll 1$ is

$$C = \exp\{-[y^2/\delta^2 + z^2/(4\mathscr{D}T)]\}, \tag{3.24}$$

where $$T = \int_{-X_p}^{x} dx'/\tilde{U}'(x') = X_p + x + \tfrac{1}{2}\ln\left(\frac{(x-1)(X_p+1)}{(x+1)(X_p-1)}\right).$$

It follows from (3.24) that the width of the plume in the y direction is the same as for a line source, so that $(\overline{N^2})^{\frac{1}{2}} = \delta/2^{\frac{1}{2}}$. If the half-width of the plume in the z direction is defined as in (3.18) then

$$(\overline{Z^2})^{\frac{1}{2}} = \mathscr{D}^{\frac{1}{2}}T. \tag{3.25}$$

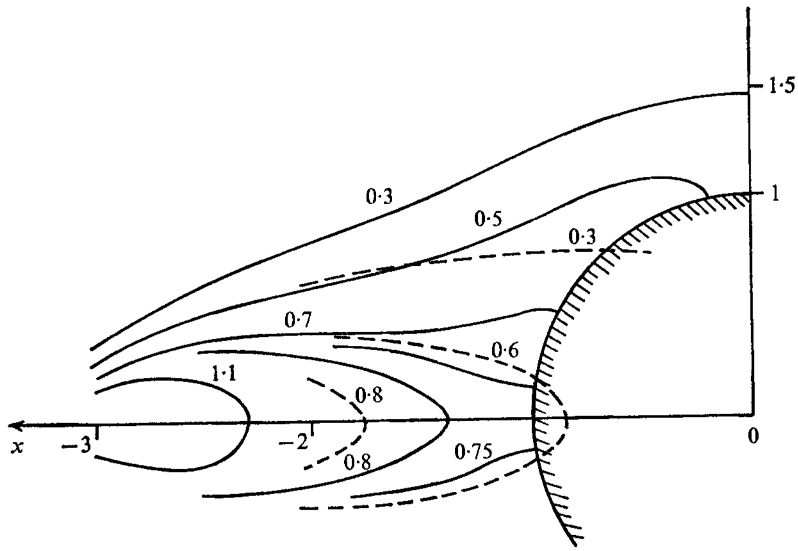


FIGURE 6. Contours of concentration for source at $(-3, 0)$. $\mathscr{D} = 0{\cdot}1$. ——, contours in presence of cylinder; --, contours in absence of cylinder.

Therefore as $x \to -1$,

$$(\overline{Z^2})^{\frac{1}{2}} \sim [-\tfrac{1}{2}\mathscr{D}\ln|x+1|]^{\frac{1}{2}},$$

which is the mathematical reason for the results shown in figures 2(*b*), 2(*c*) and 4 that the plume width also becomes very large in the *z* direction as well as the *y* direction, though it grows more slowly as the stagnation point is approached – a similar result to that obtained from the statistical analysis. The most interesting difference between the line and point sources is the value of *C*, since for the latter

$$C(y=0) \sim 1/\{4\pi\mathscr{D}(X_p + 1/X_p - 2)(-\tfrac{1}{2}\ln|x+1|)\} \quad \text{as} \quad x \to -1, \tag{3.26}$$

so that at the stagnation point (in the limit $x \to -1$, $\mathscr{D} \to 0$) $C \to 0$. This result can be explained by means of the continuity equation (3.23), which for a *point* source involves the width in the *z* direction, as well as δ:

$$C \times \delta \times \tilde{U}' \times (\overline{Z^2})^{\frac{1}{2}} = \text{constant},$$

and therefore, because

$$(\overline{Z^2}) \to \infty \quad \text{and} \quad \delta \times \tilde{U}' \to \text{constant}, \quad C \to 0 \quad \text{as} \quad x \to -1.$$

Despite this mathematical limit, when $x = -1{\cdot}01$, the ratio of C to C_0, the value of C in the absence of the body, was found to be only 0·83. In fact when

$$|x+1| = o(\mathscr{D})$$

the solution (3.21) breaks down, and a new solution (not yet found) is required.

Sources off the stagnation streamline. Equations (3.5) and (3.12) were used to calculate $(\overline{N^2})^{\frac{1}{2}}$ and C for line and point sources off the stagnation line, shown in figures 2(*b*), 2(*c*), 4 and 5. These solutions are only valid if the plumes do not touch the body surface. Once that happens (3.10) is violated. They are therefore useful for $\mathscr{D}$ not too large. The curves for $(\overline{N^2})^{\frac{1}{2}}$ in figures 2(*b*) and 4 show that the diffusion-equation solution is similar to that of the statistical analysis in that, in the x, y plane, the plume widens as the streamline approaches the stagnation-point region and then *contracts* as the flow passes over the top of the cylinder. In figure 5(*a*) it is interesting to see how in the plumes of *line* sources, despite the considerable variations in the values of $(\overline{N^2})^{\frac{1}{2}}$ for different source positions, along the line $x = -1$ the value of C only varies by 15% between the plumes emanating from sources at $(-3, 0)$ and $(-3, 1{\cdot}0)$, i.e. curves I and III. The fact that C is greatest on the stagnation line has already been discussed. Then even these small differences disappear as the streamlines converge and the flow accelerates near the top of the cylinder at $x = 0$. It is surprising to note that, on $x = 0$, C is actually slightly lower than C_0 whatever the position of the source. In figure 5(*b*) a different pattern emerges with C being *less* than C_0 for the two plumes emanating from *point* sources on or near the stagnation line (I and II). The reason is that the time for a particle to reach the line $x = -1$ from its source is greater in the presence of the cylinder and therefore $(\overline{Z^2})^{\frac{1}{2}}$ is larger. But C is *greater* than C_0 in a plume emanating from a point source placed a unit radius off the stagnation line because $(\overline{Z^2})^{\frac{1}{2}}$ is only slightly greater than in the absence of the cylinder and therefore this effect causes a smaller reduction than the two-dimensional amplification shown in figure 5(*a*).

Source at the stagnation point. Halitsky (1968, p. 246) performed an interesting experiment when examining diffusion around the model of a cylindrical reactor building. He placed a source at the upstream stagnation point and measured the dispersion of the pollutant as it was convected round the circular cylinder. To show how the diffusion equation can throw light on this complex problem, we now consider the exact two-dimensional solution for the concentration. Near the stagnation point, in our normalized co-ordinates and variables

$$\phi - \phi_p = s^2 - n^2, \quad \psi - \psi_p = 2sn,$$

where the s direction is parallel and the n direction perpendicular to the body. Then from the exact solution for the concentration, equation (3.5), one obtains

$$C = \frac{1}{2\pi\mathscr{D}} \exp\left(\frac{s^2 - n^2}{2\mathscr{D}}\right) K_0\left(\frac{s^2+n^2}{2\mathscr{D}}\right). \tag{3.27}$$

Note that $\partial C/\partial n = 0$ at $n = 0$.

Now consider the two limiting regions of the flow:

(i) As $s^2 + n^2 \to 0$,

$$C \sim -(\pi\mathscr{D})^{-1} \ln\left[(s^2+n^2)^{\frac{1}{2}}\right]. \tag{3.28}$$

That is, very close to the stagnation point, where the mean velocity is zero, the concentration is a function of radial distance from the source only.

(ii) When $(s^2+n^2)/\mathscr{D} \gg 1$,

$$C \sim \exp(-n^2/\mathscr{D}) / [4\pi\mathscr{D}(s^2+n^2)]^{\frac{1}{2}}. \tag{3.29}$$

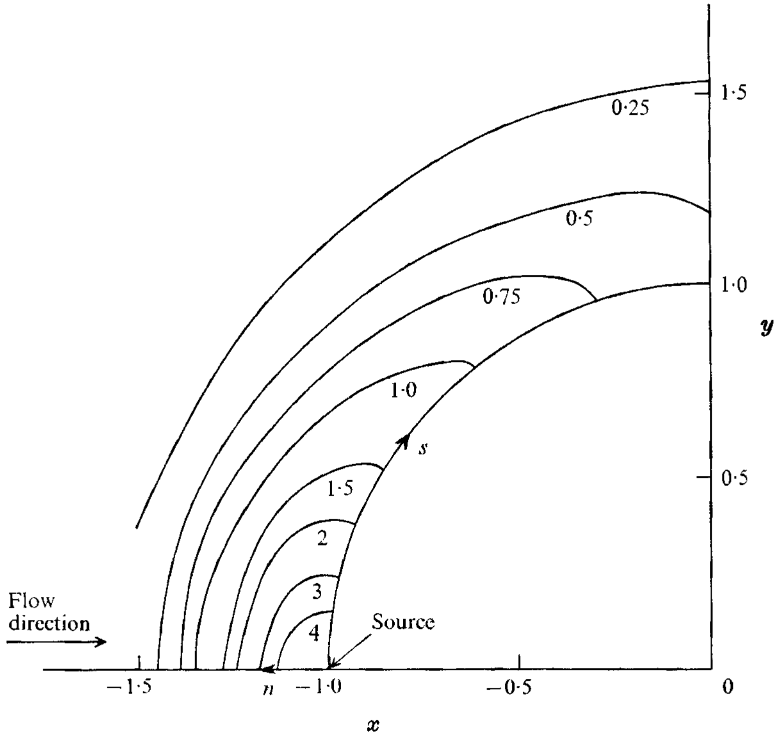


FIGURE 7. Contours of concentration for line source at $(-1, 0)$, the upstream stagnation point.

Thus when $s^2 \gg n^2$
$$C \sim \exp(-n^2/\mathscr{D})/[(4\pi\mathscr{D})^{\frac{1}{2}} s], \tag{3.30}$$
which means that, as the plume is convected downstream, first, its thickness does not increase and second, the concentration falls away like s^{-1} rather than the slower rate of $s^{-\frac{1}{2}}$ found in constant-velocity flows. Both these effects are a result of the flow accelerating along the body's surface. This case can be computed exactly from (3.5) and the results thus obtained are shown in figure 7. Here $\mathscr{D} = 0{\cdot}1$ as in figure 6. Contrary to the results presented here, Halitsky found in his experiments that the plume was thickest at $s = 0$. This was most likely due to downwash on the front of the cylinder with a vortex curling around the front of the cylinder at ground-level. Our assumption of two-dimensionality excludes such effects.

The effect of a hill. If one idealizes flow over a hill as irrotational flow over a cylindrical obstacle and assumes uniform flow upstream one can transform the x, y plane into a ϕ, ψ plane (see figure 8). If the effect of the ground is then represented by an 'image' source the boundary condition is satisfied and, for example, ground-level concentrations can be calculated. From (3.9) it follows that using an image source at $\psi = -\psi_p$ the ground-level concentration for a *line* source
$$C_g = \frac{2}{[4\pi(\phi-\phi_p)\,\mathscr{D}]^{\frac{1}{2}}} \exp\{-\psi_p^2/[4\mathscr{D}(\phi-\phi_p)]\}.$$

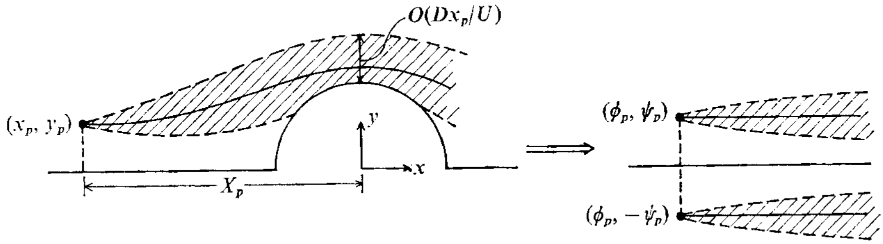


FIGURE 8. Configuration for flow around a hill in x, y and ϕ, ψ space. In ϕ, ψ space the 'image' source is shown at $(\phi_p, -\psi_p)$.

To calculate the maximum ground-level concentration $C_{g\max}$ differentiate C_g with respect to ϕ. Then

$$-\frac{1}{2(\phi-\phi_p)}+\frac{\psi_p^2}{4\mathscr{D}(\phi-\phi_p)^2}=0,$$

whence at the point of maximum ground-level concentration

$$\phi-\phi_p=\psi_p^2/2\mathscr{D}, \tag{3.31}$$

and therefore

$$C_{g\max}=1/(\tfrac{1}{2}\pi e\psi_p^2)^{\frac{1}{2}}. \tag{3.32}$$

We now take as an example of a hill a semicircle of height h, in terms of which the flow parameters are normalized, and we assume that the source has a height H and is a distance X_p upwind of the centre-line of the hill. Then

$$\psi_0 \simeq \frac{H}{h}\left(1-\frac{h^2}{H^2+X_p^2}\right),$$

whence

$$C_{g\max} \simeq \frac{(2/\pi e)^{\frac{1}{2}}}{(H/h)\,[1-h^2/(X_p^2+H^2)]}, \tag{3.33}$$

and therefore we have the interesting result that the ratio λ of $C_{g\max}$ with and without the hill is

$$\lambda=\frac{C_{g\max}(h)}{C_{g\max}(h=0)}=\frac{1}{1-h^2/(X_p^2+H^2)}. \tag{3.34}$$

Thus the maximum ground-level concentration is increased indefinitely (i.e. $\lambda \to \infty$) as the position of the source approaches the surface of the hill. For the common practical situation where the source is placed some distance upwind and $X_p \gg H$, the fractional increase in $C_{g\max}$ is $(h/X_p)^2$, which one notes is independent of H/h. The calculation would certainly suggest that the maximum ground-level concentration will usually only increase fractionally when pollutant is blown by the wind over a two-dimensional ridge of hills. (The interesting result (3.31) and (3.32), but not our general conclusions, for line sources were originally obtained by Stumke (1964), but as they are not widely known we have repeated their derivation here.) Other interesting investigations along these lines have been reported by Berlyand (1972). In figure 9(a) ground-level concentrations from a source at $x_p = -2$ with various values of H/h are shown for $\mathscr{D} = 10^{-2}$ with and

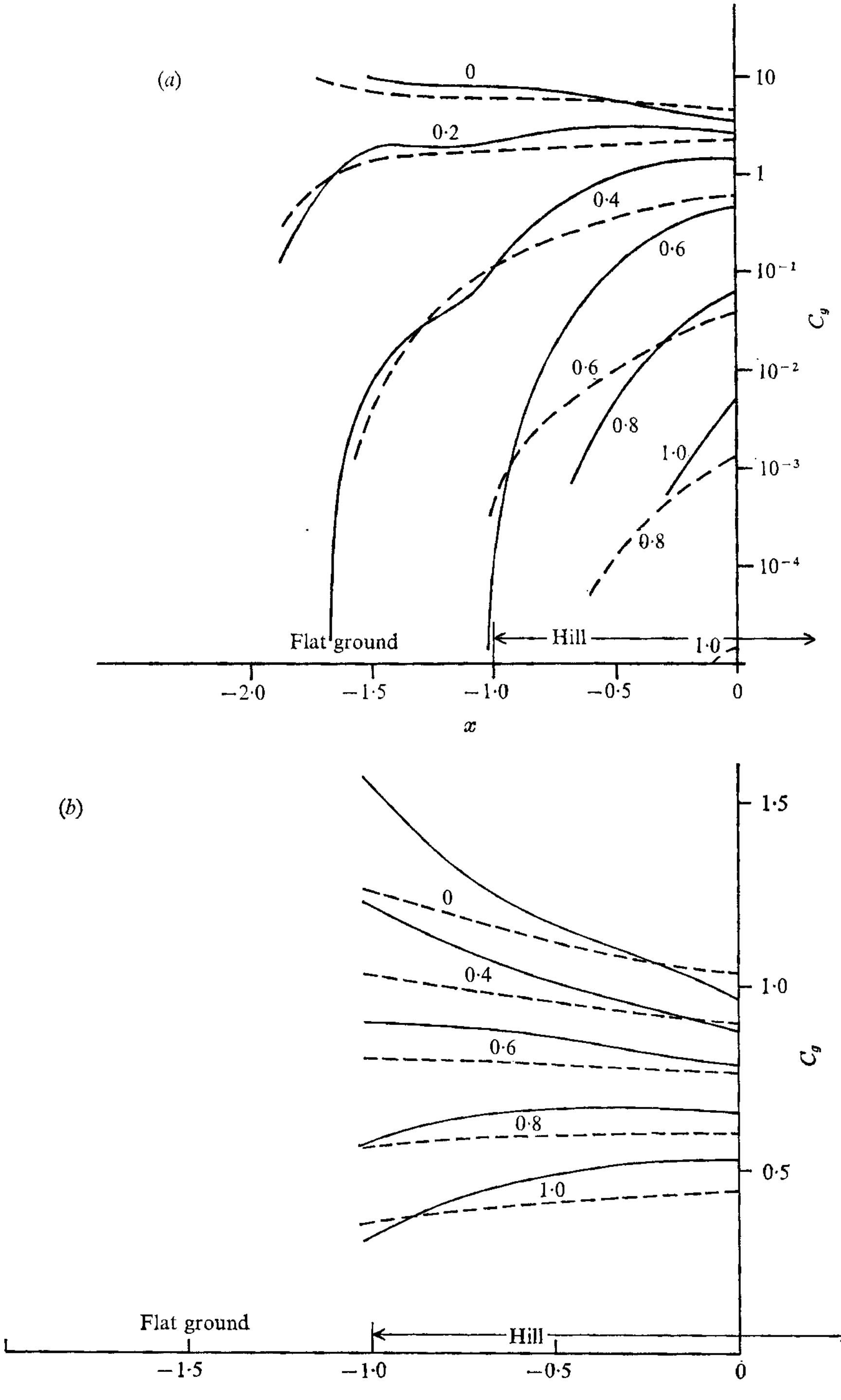


FIGURE 9. Ground-level concentrations for various source heights with and without a hill downstream. ——, concentration with hill; ––, without hill. Source heights are shown on curves for a hill of height h. (*a*) $\mathscr{D} = 0{\cdot}01$, sources placed at various heights H/h in the plane, $x_p = -2$. Values of H/h are shown on each curve. (*b*) $\mathscr{D} = 0{\cdot}1$, sources placed in the plane $x_p = -3$.

without the hill present. If we estimate $\mathscr{D}$ in terms of turbulence parameters as $\mathscr{D} = (\overline{u_{\infty 2}^2})^{\frac{1}{2}}/L_E$ and $L_E \sim h$, the hill height, this corresponds to $(\overline{u_{\infty 2}^2})^{\frac{1}{2}}/U_\infty = 1\%$. For such a low turbulence level the plume is very narrow with high concentrations near the plume centre-line. In this case a decrease in the distance between the ground and the plume centre-line, caused by the presence of a hill, leads to a large increase in local ground-level concentrations from values which are very low on flat terrain. These values are still very small however compared with those on the plume centre-line. As predicted by (3.34) the maximum ground-level concentrations do not increase significantly, and as predicted by (3.31) for $H/h > 0{\cdot}4$ these maxima must occur well downwind of the hill.

Taking next a case where the turbulence is more vigorous and $\mathscr{D} = 0{\cdot}1$, figure 9(*b*) then shows that the maximum ground-level concentration occurs upwind of or on the hill. For a source the same height as the hill we note a 20% increase in C_g in agreement with (3.34). Possibly the most significant effect of a hill is that the average concentration over the hill's surface is much greater than over an equal area of flat terrain for the same height of chimney. Where the surface absorbs pollutant, this must be a serious consideration, even though $C_{g\,\max}$ is not significantly increased.

4. Discussion

In our statistical analysis of §2 we have shown formally that, if Lagrangian autocorrelations or in limiting cases Eulerian autocorrelations in moving coordinates of the turbulent velocity are known only simple integrals, (2.10) and (2.26), need to be evaluated to find $(\overline{N^2})^{\frac{1}{2}}$ or $(\overline{Z^2})^{\frac{1}{2}}$, the effective transverse dimensions of a plume emanating from a line or point source. This in itself seems a significant step forward since the combined effects of turbulent diffusion and convection in a changing velocity field make for a very complicated problem.

In general quantitative predictions can probably only be obtained for cases where one can calculate these velocity correlations, since if the integrals were to be calculated using experimental data the amount of data required would be prohibitive. There is no method of calculating Lagrangian autocorrelations, but when $\alpha \ll 1$ and $\alpha a/L_E \ll 1$ then Hunt (1973) has shown that the 'pseudo-Lagrangian' autocorrelations defined in (2.20) can be calculated in terms of the spectrum of the upstream turbulence. It only appears to be computationally feasible to calculate the turbulence when $L_E \gg a$, the case examined in §2.2, or when $L_E \ll a$, a case which will be examined in a later paper.

The advantages of the statistical method are clear: it is based on an accurate representation of the turbulence and the results show the effect of different turbulence properties on the dispersion from a source. There are, however, some serious disadvantages. First, only $(\overline{N^2})^{\frac{1}{2}}$ and $(\overline{Z^2})^{\frac{1}{2}}$ are easily calculable, the distributions of concentration are not. Second, it is not easy to take into account turbulence properties varying between streamlines. Therefore, because the turbulent velocity normal to the body's surface is zero, the simple method cannot enable the boundary condition $\partial C/\partial n = 0$ to be satisfied. In the complicated flow near the stagnation point, even though the turbulent velocity has been calculated, its effect on diffusion has not. Third, and most important, the results are only valid if the distance x of the source upstream is such that $(\overline{N^2}(x))^{\frac{1}{2}} \ll L_E$ or a, i.e. $x \ll L_E U_\infty/(\overline{u^2_{\infty 2}})^{\frac{1}{2}}$, for very large buildings or hills, and $x \ll aU_\infty/(\overline{u^2_{\infty 2}})^{\frac{1}{2}}$, for most buildings with a dimension a.

The diffusion equation for sources placed in the potential flows we are discussing here has the great merit that, thanks to King's (1914) solution, it can be solved easily. It can be solved (*a*) for moderate values of $\mathscr{D}$ as well as small values, which means in terms of turbulent diffusion that the turbulent intensity can be high, and (*b*) in a complicated flow near a boundary such as a stagnation-point region, and (*c*) the solution gives the distribution of C, which can be quite complicated and far from a simple Gaussian profile. The disadvantage is the very serious one that it is difficult to relate $\mathscr{D}$ to the turbulence properties, even if it is considered a variable function.

In considering these two approaches we are of the opinion that once an approximate value for $\mathscr{D}$ has been found by comparison with the statistical approach the solution to most complicated, non-uniform diffusion problems may best be estimated by means of the diffusion equation. This approach may give physical insight into many diffusion problems hitherto regarded as too complicated to analyse theoretically, for example the problem of ground-level concentrations over hills, and may provide some justification for the mathematical models now being deployed to compute concentrations of air pollution in hilly terrain (e.g. Hino 1968).

We intend to extend the work begun here and to apply the conclusions to practical situations.

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